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Some New Developments in the Theory of Path Integrals, with Applications to Quantum Theory

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A survey of recent developments concerning rigorously defined infinite dimensional integrals, mainly of the type of "Feynman path integrals," is given. Both the theory and its applications, especially in quantum theory, are presented. As for the theory, general results are discussed including the case of polynomially growing phase functions, which are handled by exploiting the connection with probabilistic functional integrals. Also applications to continuous measurement theory and the stochastic Schrödinger equation are given. Other applications of probabilistic methods in non relativistic quantum theory and in quantum field theory, and their relations with statistical mechanics, are discussed.

KEY WORDS: Feynman path integrals; Schrödinger equation; euclidean quantum fields; convoluted Poisson noise; Poisson random fields.

It is a great pleasure to dedicate this paper to Gianni Jona-Lasinio on the occasion of his 70th birthday—as a small sign of deep gratitude.

The topic of the present work belongs to just one of several areas the first named author had the opportunity and pleasure to discuss with Gianni over many years, always appreciating his genuine sense for what is essential in mathematical physics and receiving from him new motivations and ideas.

The main body of this paper is Section 2, where we give a rather detailed discussion of some new developments concerning particular aspects of path integral methods in quantum theory. In Sections 1 and 3 we take the opportunity to mention some other connections with work and interests of Gianni, even though time and space do not permit to present them here in any details.[4]

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[4] These connections were discussed a bit more extensively in the lecture given by the first named author.

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1. PROBABILISTIC METHODS IN QUANTUM MECHANICS

Let us take as starting point the connection between quantum dynamics, as given by a family of unitary operators $U_t = e^{-itH}$, $t \in \mathbb{R}$ being the time variable and H the (lower semibounded) Hamiltonian, and the corresponding heat equation evolution semigroup $U_\tau = e^{-\tau H}$, $\tau \geqslant 0$. Heuristically, but also rigorously under some assumptions on H, U_τ is the analytic continuation of U_t for "imaginary time" (i.e., for t replaced by $-i\tau$). The corresponding functional representation for H of the Schrödinger form $-\frac{\Delta}{2}+V$ in $L^2(\mathbb{R}^d, dx)$, with a suitable real-valued potential V, is, for the heat equation, given by the Feynman–Kac formula

$$U_\tau f(x) = \int e^{-\int_0^\tau V(w(s))\,ds} f(w(\tau))\, P^x(dw), \tag{1}$$

P^x being Wiener measure giving the distribution of the Wiener (or Brownian motion) process $(w(s), 0 \leqslant s \leqslant t)$ started at $x \in \mathbb{R}^d$ at time 0, f the initial condition. Let us stress that (1) is a well defined integral (e.g., for V, f bounded and continuous.

Heuristically we have

$$e^{-\int_0^\tau V(w(s))\,ds} P^x(dw) = \text{“}Z^{-1} e^{-S_\tau^E(w)}\, Dw\text{”} \equiv P_E^x(dw), \tag{2}$$

with S_τ^E the euclidean action functional,

$$S_\tau^E(w) = S_\tau^0(w) + \int_0^\tau V(w(s))\,ds, \qquad S_\tau^0(w) \equiv \tfrac{1}{2}\int_0^\tau |\dot{w}(s)|^2\,ds \tag{3}$$

Z is the normalization ("heuristic free partition function")

$$Z = \text{“}\int e^{-S_\tau^0(w)}\, Dw\text{”} \tag{4}$$

with Dw a "flat measure" (the E in P_x^E stands for "euclidean"). One writes

$$U_\tau f(x) = \mathbb{E}_{P_E^x} f(\omega(\tau)) \tag{5}$$

(with $\mathbb{E}_Q \equiv$ expectation with respect to the measure Q). We can write, in the sense of linear functionals

$$U_\tau f(x) = \langle f(\cdot), P_E^x(\cdot)\rangle \tag{6}$$

($\langle\ ,\ \rangle$ being the dualization between a space of function f, say $C_b(\Omega)$, and the space of finite measures on $\Omega \equiv C([0, t]; \mathbb{R})$).

Similarly, for the solutions of the corresponding Schrödinger evolution one can expect, following Feynman (see Section 2), a formula of the type

$$U_t f(x) = \langle f(\cdot), P^x_F(\cdot)\rangle, \tag{7}$$

where the r.h.s. has to be understood as the evaluation at f of a (linear continuous) "Feynman path functional" (usually called Feynman path integral). P^x_F (where F stands for "Feynman") is heuristically obtained by replacing the exponents $e^{-S^E_\tau(w)}$ in (2) and $e^{-S^0_\tau(w)}$ in (4) with $e^{\frac{i}{\hbar}S_t(w)}$ and $e^{\frac{i}{\hbar}S^0_t(w)}$ respectively, where S_t is the "classical action functional" defined as S^E_t in (3) but with V replaced by $-V$.

Whereas Feynman–Kac formula (1) holds essentially for any V having negative part with not too strong singularities and H lower bounded, the above Feynman formula (7), holds essentially for any continuous V (not necessarily such that H is lower bounded), see Section 2 for a further discussion of Feynman path integrals. Let us remark that for historical reasons (to conform to the original presentation of Feynman) (see, e.g., ref. 44) (7) is usually rewritten, using the invariance of $P^x_F(d\omega)$ under the transformation induced in Ω by $s \mapsto \tau - s$, as $U_t f(x) = \langle f(\cdot), \tilde{P}^x_F(\cdot)\rangle$, with $\tilde{P}^x_F$ defined as P^x_F, but with the space of paths replaced by the one where the paths end at time t at x (and start anywhere at the initial time zero). In Section 2 we shall use this representation and denote the paths by γ in order not to confuse them with the previous paths.

One way to understand the principle of this, is to realize that for a large class of V (see, e.g., ref. 69), H, as an operator on $L^2(\mathbb{R}^d, dx)$, has a strictly positive eigenfunction ϕ in $L^2(\mathbb{R}^d, dx)$, called "ground state," such that $H \geqslant c$, for some $c > -\infty$, and $H\phi = c\phi$. Then $(H-c)$ is unitary equivalent to a self-adjoint positive operator H_μ in $L^2(\mathbb{R}^d, \mu)$, with $\mu(dx) \equiv \phi^2(x)\, dx$, the unitary equivalence being given by $f \in L^2(\mathbb{R}^d, dx) \leftrightarrow \frac{f}{\phi} \in L^2(\mathbb{R}^d, \mu)$, so that $e^{-it(H-c)}f = \phi e^{-itH_\mu}\phi^{-1} f$. A simple computation shows that H_μ is the self-adjoint operator uniquely associated with the closed sesquilinear form $\mathscr{E}_\mu(f, g) \equiv \frac{1}{2}\int \nabla\bar{f}\, \nabla g\, d\mu$, $f, g \in D(\nabla)$, ∇ being the natural gradient operator from $L^2(\mathbb{R}^d, \mu)$ to $L^2(\mathbb{R}^d, \mu) \otimes \mathbb{R}^d$ (see, e.g., ref. 2). $\mathscr{E}_\mu$ is the natural extension to the complex $L^2(\mathbb{R}^d, \mu)$ of the classical Dirichlet form given by μ. Under some weak regularity assumptions on ϕ (which, can be reinterpreted in terms of regularity assumptions on V), there exists a measurable "drift vector," β_μ (on $\mathbb{R}^d$ with values in $\mathbb{R}^d$), such that $H_\mu = -\frac{1}{2}\Delta - \beta_\mu \nabla$ (on a dense domain) in $L^2(\mathbb{R}^d, \mu)$. By the theory of Dirichlet forms the corresponding semigroup $(e^{-\tau H_\mu})_{\tau \geqslant 0}$ is the transition semigroup of a diffusion process $X(\tau)$, $\tau \geqslant 0$ on $\mathbb{R}^d$, satisfying a stochastic differential equation of Langevin's type: $dX(\tau) = \beta_\mu(X(\tau))\, d\tau + dw(\tau)$, with

$(w(s), s \in [0, \tau])$ a standard Brownian motion on $\mathbb{R}^d$. The initial distribution can be taken to be any point in $\mathbb{R}^d$, μ is an invariant measure for $X(\tau)$. One has in particular for any $g \in C_b(\mathbb{R}^d)$:

$$(e^{-\tau H_\mu} g)(x) = \mathbb{E}^x_{|X}(g(X(\tau))) = \mathbb{E}^x[e^{-\frac{1}{2}\int_0^\tau \beta(w(s))^2\,ds} e^{-\int_0^\tau \beta(w(s))\,dw(s)} g(w(\tau))]$$

(with $\mathbb{E}^x_{|X}$ the expectation with respect to the distribution of X started at x and $\mathbb{E}^x$ the one with respect to the Wiener measure P^x). The density $e^{-\int_0^\tau \beta(w(s))^2\,ds} e^{-\int_0^\tau \beta(w(s))\,dw(s)}$ with respect to the Wiener measure $P^x(dw)$ is called Girsanov functional. Combining this formula with

$$e^{-\tau H_\mu} = \phi^{-1} e^{-\tau(H-c)} \phi$$

we also obtain

$$(e^{-\tau H_\mu} g)(x) = \phi^{-1}(x)\, \mathbb{E}^x[e^{-\int_0^\tau V(w(s))\,ds} \phi(w(\tau))\, g(w(\tau))].$$

These are just a few examples of transformation formulae which constitutes the basis of stochastic analysis. In particular the representations relating the Dirichlet operator H_μ and X extend to more general situations, where the state space and the coefficients can be "singular" and the state space $\mathbb{R}^d$ is replaced by some possibly infinite-dimensional state space. See, e.g., refs. 1, 2, 54, and references therein for these connections. The work of Jona-Lasinio and his coworkers has played an essential role in pointing out the relations between processes like the above X and quantum theory providing new insights in vast areas of stochastic analysis and quantum theory (in the non relativistic theory as well as the relativistic one, in particular in connection with Nelson's quantum mechanics and the Euclidean approach, see also Section 3). The latter has also permitted to exploit probabilistic as well as classical statistical mechanical methods in quantum (field) theory). Let us point out that stochastic processes have been used in quantum theory both in an instrumental way (e.g., for deriving spectral estimates[54, 55, 71]) and in a conceptual way (e.g., in stochastic mechanics or in the theory of Schrödinger processes, see, e.g., refs. 1–49, 52, 69, 70, 77, and references therein).

2. SOME BASIC ELEMENTS AND NEW DEVELOPMENTS IN THE THEORY OF FEYNMAN PATH INTEGRALS

The representation (7) in Section 1 of the solution of Schrödinger equation was already presented heuristically by Feynman in his thesis in the forties, following a suggestion by Dirac. Let us rewrite it with the (reduced) Planck's constant $\hbar$ inserted and with ψ_0 instead of f for initial condition. Schrödinger equation is then

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta\psi + V\psi \\ \psi(0, x) = \psi_0(x) \end{cases} \tag{8}$$

(7) is understood by Feynman as a way to compute the wave function ψ at time t evaluated at the point $x \in \mathbb{R}^d$ as an "integral over histories," that is as an integral over all possible paths with finite energy arriving at time t at the point x:

$$\psi(t, x) = \text{``} \int_{\{\gamma \mid \gamma(t) = x\}} e^{\frac{i}{\hbar} S_t(\gamma)} \psi_0(\gamma(0))\, D\gamma \text{''} \tag{9}$$

$S_t(\gamma)$ being the classical action (introduced in Section 1) evaluated along the path γ, and $D\gamma$ an heuristic "flat" measure on the space of paths. We remark that (7) resp. (9) are entirely similar to (6) resp. (5) (taking into account (2)). In (9) we write γ instead of ω as used in (5), just to stress that we are in the framework of Feynman rather than in the one of probabilistic integrals.

Even if more than 50 years have passed, Feynman's formula is still fascinating, as it shows the link between the classical description of the physical world and the quantum one. In fact it provides a quantization method, allowing to associate, at least heuristically, a quantum evolution to each classical Lagrangian. Moreover it allows the study of the "semi-classical limit," that is the study of the behavior of the solution of the Schrödinger equation taking into account that $\hbar$ is small. Indeed since $\hbar$ is small the integrand is strongly oscillating and the main contribution to the integral should come from those paths γ which make stationary the phase function S. These, by Hamilton's least action principle, are the classical orbits of the system.

Formula (9), as it stands, has not of course a well defined mathematical meaning. Indeed neither the normalization constant, nor the "flat measure" $D\gamma$ on the space of paths are well defined. Formula (1) in Section 1 was derived by Kac in 1949 precisely as a rigorous replacement, valid for the corresponding heat equation, of the heuristic expression (9). As we already mentioned in Section 1, (1) is a well defined integral on the space of continuous paths with respect to a σ-additive measure P^x (for paths starting at x). Such an interpretation is not possible for the heuristic "Feynman measure" $e^{\frac{i}{\hbar} S_t(\gamma)} D\gamma$. Indeed Cameron[33] proved that the latter cannot be realized as a complex σ-additive measure, even on very nice subsets.

Nevertheless, under suitable hypothesis on the potential V and on the initial datum ψ_0, one can indeed give to the integral (9) a rigorous mathematical meaning as a linear continuous functional on a suitable class of functions, as is expressed by (7) (whose probabilistic counterpart is (6)). In the literature several realizations of the "Feynman functional" can be found, for instance by means of analytic continuation,[33, 35, 41, 52, 60, 64, 66, 67, 73, 74] by non standard analysis[7] or as an infinite dimensional distribution in the framework of Hida calculus,[40, 49] or via "complex Poisson measures,"[1, 65] or as a infinite dimensional oscillatory integral.[4, 18, 42, 50, 51] The latter method is particularly interesting as it is the only one by which a development of an infinite dimensional version of the stationary phase method and the corresponding study of the semiclassical limit ($\hbar \downarrow 0$) of the solution has been performed.

In the following we shall denote by $\mathcal{H}$ a (finite or infinite dimensional) real separable Hilbert space, whose elements, resp. scalar product, will be denoted by $x, y \in \mathcal{H}$, resp by $\langle x, y \rangle$. $f: \mathcal{H} \to \mathbb{C}$ will be a function on $\mathcal{H}$ and $Q: D(Q) \subseteq \mathcal{H} \to \mathcal{H}$ an invertible, densely defined and self-adjoint operator.

We shall denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures on $\mathcal{H}$, endowed with the total variation norm, that is:

$$\mu \in \mathcal{M}(\mathcal{H}), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\bigcup_i E_i = \mathcal{H}$. $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E - x)\, \nu(dx), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H})$$

and the unit element is the vector δ_0.

We will denote by $\mathcal{F}(\mathcal{H})$ the space of complex functions on $\mathcal{H}$ which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, that is:

$$f: \mathcal{H} \to \mathbb{C} \qquad f(x) = \int_H e^{i\langle x, \beta \rangle} \mu_f(d\beta) \equiv \hat{\mu}_f(x).$$

$\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one, the unit element is the function 1, i.e., $1(x) = 1 \ \forall x \in \mathcal{H}$ and the norm is given by $\|f\| = \|\mu_f\|$.

Let us assume first of all that $\mathscr{H}$ is finite dimensional, i.e., $\mathscr{H} = \mathbb{R}^n$, and define the "Fresnel integral"

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, dx$$

in the following way

Definition 1. A function $f: \mathbb{R}^n \to \mathbb{C}$ is Fresnel integrable with respect to Q if and only if for each $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the limit

$$\lim_{\epsilon \to 0} (2\pi i\hbar)^{-n/2} \int e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, \phi(\epsilon x)\, dx \tag{10}$$

exists and is independent of ϕ. In this case the limit is called the Fresnel integral of f with respect to Q and denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, dx. \tag{11}$$

The description of the full class of Fresnel integrable functions is not easy, but one can find some interesting subsets of it. Indeed the following result holds:[18, 42]

Theorem 1. Let $f \in \mathscr{F}(\mathbb{R}^n)$. Then f is Fresnel integrable and its Fresnel integral with respect to Q is given by:

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, dx = (\det Q)^{-1/2} \int e^{\frac{-i\hbar}{2}\langle \alpha, Q^{-1}\alpha\rangle} \mu_f(d\alpha). \tag{12}$$

The definition of oscillatory integral can be extended to the case where $\mathscr{H}$ is infinite dimensional.[4, 42]

Definition 2. A function $f: \mathscr{H} \to \mathbb{C}$ is Fresnel integrable with respect to Q if and only if for each sequence P_n of projectors onto n-dimensional subspaces of $\mathscr{H}$, such that $P_n \leqslant P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$, (1 being the identity operator in $\mathscr{H}$), the finite dimensional approximations of the Fresnel integral of f with respect to Q

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathscr{H}} e^{\frac{i}{2\hbar}\langle P_n x, QP_n x\rangle} f(P_n x)\, d(P_n x),$$

are well defined and the limit

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, QP_n x\rangle} f(P_n x)\, d(P_n x) \tag{13}$$

exists and is independent of the sequence $\{P_n\}$.

In this case the limit is called the Fresnel integral of f with respect to Q and is denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, dx.$$

One can prove[4, 42] that if $f \in \mathcal{F}(\mathcal{H})$ then $f \circ P_n \in \mathcal{F}(P_n(\mathcal{H}))$ and f is Fresnel integrable. Moreover, if $Q-I$ is trace class, the following Cameron-Martin–Parseval type formula holds:

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x, Qx\rangle} f(x)\, dx = (\det Q)^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha, Q^{-1}\alpha\rangle} \mu_f(d\alpha) \tag{14}$$

where $\det Q = |\det Q|\, e^{-\pi i \,\mathrm{Ind}\, Q}$ is the Fredholm determinant of the operator Q, $|\det Q|$ is its absolute value and $\mathrm{Ind}(Q)$ is the number of negative eigenvalues of the operator Q, counted with their multiplicity.

In this setting one can give a rigorous mathematical interpretation of formula (9) in terms of an infinite dimensional oscillatory integral on a suitable Hilbert space of paths. Let us consider the Sobolev space $\mathcal{H} = H^{1,2}_{(t)}([0,t], \mathbb{R}^d)$, that is the space of absolutely continuous functions $\gamma: [0,t] \to \mathbb{R}^d$, $\gamma(t) = 0$, such that $\int_0^t |\dot\gamma(s)|^2\, ds < \infty$, endowed with the following scalar product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot\gamma_1(s) \cdot \dot\gamma_2(s)\, ds.$$

$\mathcal{H}$ is essentially the "Cameron-Martin space."

Let us consider, on the other hand, the Schrödinger equation in $L^2(\mathbb{R}^d)$

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \tag{15}$$

with initial datum $\psi_{|t=0} = \psi_0$, where $H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2} x\Omega^2 x + V'(x)$, where $x \in \mathbb{R}^d$, $\Omega^2 \geqslant 0$ is a $d \times d$ matrix, $V' \in \mathcal{F}(\mathbb{R}^d)$ and $\psi_0 \in \mathcal{F}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ (we set $m = 1$ for simplicity of notation).

By considering the operator L on $H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$ given by

$$\langle \gamma, L\gamma\rangle \equiv \int_0^t \gamma(s)\,\Omega^2\gamma(s)\,ds,$$

and the function $W\colon H_t \to \mathbb{C}$

$$W(\gamma) \equiv \int_0^t V'(\gamma(s)+x)\,ds + 2x\Omega^2 \int_0^t \gamma(s)\,ds, \qquad \gamma \in H^{1,2}_{(t)},$$

formula (9)

$$\text{“const} \int_{\{\gamma \mid \gamma(t)=x\}} e^{\frac{i}{\hbar}\int_0^t (\frac{1}{2}\dot\gamma(s)^2 - \frac{1}{2}\gamma(s)\,\Omega^2\gamma(s) - V'(\gamma(s)))\,ds}\,\psi_0(\gamma(0))\,D\gamma\text{”}$$

can be interpreted as the rigorously defined infinite dimensional oscillatory integral on $H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle \gamma, (I+L)\gamma\rangle} e^{-\frac{i}{\hbar}W(\gamma)}\,\psi_0(\gamma(0)+x)\,d\gamma. \tag{16}$$

Moreover one can prove[4, 42] that (16) is a representation of the solution of (15) evaluated in $x \in \mathbb{R}^d$ at time t.

2.1. Application to the Quantum Theory of Measurement

It is possible to extend the definition of infinite dimensional oscillatory integral in order to consider complex-valued phase functions. This allows to give Feynman path integrals representations to a larger class of Schrödinger equations, with applications to the quantum theory of measurement (see refs. 13 and 14).

Theorem 2. Let $\mathcal{H}$ be a real separable Hilbert space, let $l \in \mathcal{H}$ be a vector in $\mathcal{H}$ and let L_1 and L_2 be two self-adjoint, trace class commuting operators on $\mathcal{H}$ such that $I+L_1$ is invertible and L_2 is non negative. Let moreover $f\colon \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$:

$$f(\gamma) = \hat\mu_f(\gamma), \qquad f(\gamma) = \int_{\mathcal{H}} e^{i\langle \gamma, \alpha\rangle}\mu_f(d\alpha).$$

Then the infinite dimensional oscillatory integral (with complex phase)

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} f(\gamma)\, d\gamma$$

is well defined and it is given by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} f(\gamma)\, d\gamma = \det(I+L)^{-1/2} \int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle\alpha - il,(I+L)^{-1}(\alpha - il)\rangle} \mu_f(d\alpha) \tag{17}$$

(L being the operator on the complexification $\mathcal{H}^{\mathbb{C}}$ of the real Hilbert space $\mathcal{H}$ given by $L = L_1 + iL_2$).

Equation (17) can be recognized as an analytic continuation of the Cameron-Martin–Parseval formula (14).

The latter theorem can be applied to the solution of a particular stochastic Schrödinger equation describing the continuous non-demolition measurement of the position of a quantum particle: the Belavkin equation.[5] It is well known that the continuous time evolution described by the ordinary Schrödinger equation is valid if the quantum system is "undisturbed," but if it is submitted to the measurement of one of its observables and interacts with the measuring apparatus this is no longer true. Indeed the state of the system after the measurement is the result of a random and discontinuous change: the so-called "collapse of the wave function." By modeling the measuring apparatus with a one-dimensional bosonic field, by assuming the interaction Hamiltonian has a particular form and by means of the quantum stochastic calculus of Hudson and Parthasarathy, in 1987 V. P. Belavkin[(30)] proposed a stochastic partial differential equation describing the selective dynamics of a d-dimensional particle submitted to the measurement of one of its (possible M-dimensional vector) observables, described by the self-adjoint operator R on $L^2(\mathbb{R}^d)$

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar} H\psi(t,x)\, dt - \frac{\lambda}{2} R^2\psi(t,x)\, dt + \sqrt{\lambda}\, R\psi(t,x)\, dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d \end{cases} \tag{18}$$

where H is the quantum mechanical Hamiltonian, W is an M-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $dW(t)$ is the Ito differential and $\lambda > 0$ is a coupling constant, which is proportional to the accuracy of the measurement. In the particular case of the description of

[5] Related equations have been proposed also by other physicists, for instance in work by Diosi, Ghirardi, Rimini and Weber, Gisin, Mensky (see the references in refs. 13 and 14).

the continuous measurement of position one has that R is the operator multiplication by x, so that Eq. (18) assumes the following form:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar} H\psi(t,x)\,dt - \frac{\lambda}{2} x^2\psi(t,x)\,dt + \sqrt{\lambda}\, x\psi(t,x)\,dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases} \tag{19}$$

while in the case of momentum measurement, (where $R = -i\hbar\nabla$) one has:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar} H\psi(t,x)\,dt + \frac{\lambda\hbar^2}{2} \Delta\psi(t,x)\,dt - i\sqrt{\lambda}\,\hbar\,\nabla\psi(t,x)\,dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d. \end{cases} \tag{20}$$

Even before Belavkin, M. B. Mensky had proposed a heuristic formula for the selective dynamics of a particle whose position is continuously observed. According to Mensky the state of the particle at time t if the observed trajectory is the path $[a]$ is given by the "restricted path integrals"

$$\psi(t,x,[a]) = \text{“} \int_{\{\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-\lambda\int_0^t (\gamma(s)-a(s))^2\,ds} \phi(\gamma(0))\,D\gamma \text{”}. \tag{21}$$

One can see that, as an effect of the correction term $e^{-\lambda\int_0^t (\gamma(s)-a(s))^2\,ds}$ due to the measurement, the paths γ giving the main contribution to the integral (21) are those closer to the observed trajectory $[a]$. By means of Theorem 2 one can prove a Feynman path integral representation to the solution of Belavkin equation and give a rigorous mathematical meaning to Mensky's heuristic formula. Indeed in the case of position measurement, the following holds:[13]

Theorem 3. Let V and ϕ be Fourier transform of finite complex measures on $\mathbb{R}^d$. Then there exists a solution of the stochastic Schrödinger equation (19) and it can be represented by the following infinite dimensional oscillatory integral with complex phase on the Hilbert space $\mathcal{H} = H^{1,2}_{(t)}([0,t],\mathbb{R}^d)$

$$\begin{aligned} \psi(t,x) &= \widetilde{\int} e^{\frac{i}{\hbar}S_t(\gamma) - \lambda\int_0^t (\gamma(s)+x)^2\,ds} \\ &\qquad \cdot e^{\int_0^t \sqrt{\lambda}(\gamma(s)+x)\,dW(s)\,ds} \phi(\gamma(0)+x)\,d\gamma \\ &= e^{-\lambda|x|^2 t + \sqrt{\lambda}\,x\cdot\omega(t)} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-2\lambda\hbar\int_0^t x\cdot\gamma(s)\,ds} \\ &\qquad \cdot e^{-i\int_0^t V(x+\gamma(s))\,ds} \phi(\gamma(0)+x)\,d\gamma \end{aligned}$$

where $l \in \mathcal{H}$, $l(s) = \sqrt{\lambda} \int_s^t \omega(\tau)\, d\tau$ and

$$L: \mathcal{H}^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}, \qquad \langle \gamma_1, L\gamma_2 \rangle = -2i\lambda\hbar \int_0^t \gamma_1(s)\, \gamma_2(s)\, ds.$$

In the case of the Belavkin equation describing momentum measurement the stochastic term plays the role of a complex random potential depending on the momentum of the particle. In this case one has to use a "phase space Feynman path integral," that is an infinite dimensional oscillatory integral on the Hilbert space $H^{1,2}_{(t)}([0,t], \mathbb{R}^d) \times L^2([0,t]) \equiv \mathcal{H} \times L_t$ (see refs. 12 and 14).

Theorem 4. Let V and ϕ be Fourier transform of finite complex measures on $\mathbb{R}^d$. Then there exist a solution of the Cauchy problem (20), which can be represented by the following "phase space Feynman path integral:"

$$\begin{aligned}\psi(t,0,\omega) &= \widetilde{\int}_{\mathcal{H}\times L_t} e^{\frac{i}{\hbar}\int_0^t (p(s)\,\dot q(s) - \frac{p^2(s)}{2})\, ds}\, e^{-\lambda \int_0^t p(s)^2\, ds} \\ &\quad \cdot e^{-\frac{i}{\hbar}\int_0^t V(q(s))\, ds}\, e^{\sqrt{\lambda}\int_0^t p(s)\, dW(s)} \phi(q(0))\, dq\, dp \\ &= \widetilde{\int}_{H_t \times L_t} e^{\frac{i}{\hbar}\langle (q,p), A(q,p)\rangle} e^{\langle (q,p), l\rangle} e^{-\frac{i}{\hbar}\int_0^t V(q(s))\, ds} \phi(q(0))\, dq\, dp \end{aligned} \tag{22}$$

where by (q,p) we denote a generic element of $\mathcal{H} \times L_t$, $q \in \mathcal{H}$, $p \in L_t$, A is the operator on the complexification of $\mathcal{H} \times L_t$ given by

$$\langle (q,p), A(q,p)\rangle = 2\int_0^t \left(p(s)\, \dot q(s)\, ds - (1 - 2i\lambda\hbar) \int_0^t p^2(s) \right) ds$$

and $l \in H_t \times L_t$.

2.2. Extension of the Theory of Feynman Path Integrals to the Case of Polynomially Growing Phase Functions

The examples discussed in 3.1 show that the infinite dimensional oscillatory integrals are a powerful tool, with application to a larger class of (deterministic resp. stochastic) Schrödinger equations. The main restriction of such integrals has been so far the fact that the potentials V for which a Feynman path integral representation of the solution of the corresponding Schrödinger equation can be defined are of the type "quadratic plus bounded perturbation."[4, 18, 42] Indeed, in order to define the infinite dimensional oscillatory integral, the correction V' to the harmonic oscillator potential $\frac{1}{2} x\Omega^2 x$ has to belong to $\mathcal{F}(\mathbb{R}^d)$, so that it is bounded. An extension to unbounded potentials which are Laplace transforms of bounded measures has been developed in refs. 6 and 61. It includes some exponentially growing potentials but does not cover the case of potentials which are polynomials of degree larger than 2. In fact the problem is not simple, as it has been proved[75] that in one dimension, if the potential is time independent and super-quadratic in the sense that $V(x) \geqslant C(1+|x|)^{2+\epsilon}$ at infinity, $C>0$ and $\epsilon>0$, then the fundamental solution of time dependent Schrödinger equation is nowhere C^1. In refs. 20 and 21 a solution to the problem of providing a direct rigorous Feynman path integral definition for such potentials (without going to the tool of analytic continuation from a representation of the heat equation as for instance in refs. 31, 41, and 67) is given and a Feynman path integral representation for the solution of the Schrödinger equation for an anharmonic oscillator potential $V(x)=\frac{1}{2} x\Omega^2 x+\lambda x^4$, $\lambda>0$, is developed. The first step is the definition and the computation of the oscillatory integral $\int_{\mathcal{H}} e^{\frac{i}{\hbar}\Phi(x)} f(x)\, dx$, when $\mathcal{H}$ is finite dimensional and the phase function $\Phi(x)=P(x)$ is an arbitrary even polynomial with positive leading coefficient. In this case a generalization of Theorem 1 can be proved. The main tool is the following lemma, which can be proved by using the analyticity of $e^{kz+\frac{i}{\hbar}P(z)}$, $z\in\mathbb{C}$, and a change of integration contour (see ref. 20).

Lemma 1. Let $P\colon \mathbb{R}^N \to \mathbb{R}$ be an even polynomial with positive leading coefficient. Then the Fourier transform of the distribution $e^{\frac{i}{\hbar}P(x)}$:

$$\tilde{F}(k)=\int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)}\, dx, \qquad \hbar\in\mathbb{R}\setminus\{0\} \tag{23}$$

is an entire bounded function and admits the following representation:

$$\tilde{F}(k)=e^{iN\pi/4M}\int_{\mathbb{R}^N} e^{ie^{i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)}\, dx, \qquad \hbar>0 \tag{24}$$

or

$$\tilde{F}(k)=e^{-iN\pi/4M}\int_{\mathbb{R}^N} e^{ie^{-i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{-i\pi/4M}x)}\, dx, \qquad \hbar<0. \tag{25}$$

Theorem 5. Let $f\in\mathcal{F}(\mathbb{R}^N)$, $f=\hat{\mu}_f$. Then the generalized Fresnel integral

$$I(f)\equiv\int e^{\frac{i}{\hbar}P(x)} f(x)\, dx, \qquad \hbar\in\bar{D}\setminus\{0\}$$

is well defined and it is given by the formula of Parseval's type:

$$\int e^{\frac{i}{\hbar}P(x)} f(x)\,dx = \int \tilde{F}(k)\,\mu_f(dk), \tag{26}$$

where $\tilde{F}(k)$ is given by (24) or (25)

$$\tilde{F}(k) = \int e^{ikx} e^{\frac{i}{\hbar}P(x)}\,dx.$$

The integral on the r.h.s. of (26) is absolutely convergent (hence it can be understood in Lebesgue sense).

It is particularly interesting to examine the case in which $P(x) = \frac{1}{2}x(I-B)\,x - \lambda A(x,x,x,x)$, where $\hbar > 0$, $\lambda \leqslant 0$, I, B are $N \times N$ matrices, I being the identity, $(I-B)$ is symmetric and strictly positive and $A\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is completely symmetric and positive fourth order covariant tensor on $\mathbb{R}^N$.

Lemma 2. Under the assumptions above the Fourier transform of the distribution $\frac{e^{\frac{i}{2\hbar}x\cdot(I-B)\,x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}A(x,x,x,x)}$:

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik\cdot x} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)\,x}}{(2\pi i\hbar)^{N/2}} e^{\frac{-i\lambda}{\hbar}A(x,x,x,x)}\,d^N x \tag{27}$$

is a bounded complex-valued entire function on $\mathbb{R}^N$ admitting the following representation

$$\begin{aligned}\tilde{F}(k) &= \int_{\mathbb{R}^N} e^{ie^{i\pi/4}k\cdot x} \frac{e^{-\frac{1}{2\hbar}x\cdot(I-B)\,x}}{(2\pi\hbar)^{N/2}} e^{\frac{i\lambda}{\hbar}A(x,x,x,x)}\,d^N x \\ &= \mathbb{E}[e^{ie^{i\pi/4}k\cdot x} e^{\frac{i\lambda}{\hbar}A(x,x,x,x)} e^{\frac{1}{2\hbar}x\cdot Bx}]\end{aligned} \tag{28}$$

where $\mathbb{E}$ denotes the expectation value with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance operator $\hbar I$.

Theorem 6 (Parseval equality). Let $f \in \mathcal{F}(\mathbb{R}^N)$, $f = \hat{\mu}_f$. Then, under the assumptions above, the generalized Fresnel integral

$$I(f) \equiv \widetilde{\int} e^{\frac{i}{2\hbar}x\cdot(I-B)\,x} e^{\frac{-i\lambda}{\hbar}A(x,x,x,x)} f(x)\,dx$$

is well defined and it is given by:

$$\widetilde{\int} e^{\frac{i}{2\hbar} x\cdot(I-B)\,x} e^{\frac{-i\lambda}{\hbar} A(x,x,x,x)} f(x)\,dx = \int \tilde{F}(k)\,\mu_f(dk), \tag{29}$$

where $\tilde{F}(k)$ is given by Eq. (28). Moreover if μ_f is such that $\forall x \in \mathbb{R}^N$ the integral $\int e^{-\frac{\sqrt{2}}{2}kx}\,|\mu_f|\,(dk)$ is convergent and the positive function $g\colon \mathbb{R}^n \to \mathbb{R}$, defined by $g(x) = e^{\frac{1}{2\hbar}x\cdot Bx} \int e^{-\frac{\sqrt{2}}{2}kx}\,|\mu_f|\,(dk)$ is summable with respect to the centered Gaussian measure on $\mathbb{R}^N$ with covariance $\hbar I$, then f extends to an analytic function on $\mathbb{C}^N$ and the corresponding generalized Fresnel integral is given by:

$$\widetilde{\int_{\mathbb{R}^N}} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)\,x}}{(2\pi i\hbar)^{N/2}}\, e^{\frac{-i\lambda}{\hbar}P(x)} f(x)\,dx = \mathbb{E}[e^{\frac{i\lambda}{\hbar}P(x)} e^{\frac{1}{2\hbar}x\cdot Bx} f(e^{i\pi/4}x)]. \tag{30}$$

Remark 1. In ref. 20 general polynomial phase functions are also discussed and analytic resp. Borel summable expansions of (24) in powers of $\hbar$ are proved

The result in Theorem 6 has been generalized to the infinite dimensional case.

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, with inner product $\langle\ ,\ \rangle$ and norm $|\ |$. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{\hbar}{2}|x|^2}$. Let $\|\ \|$ be a "measurable" norm on $\mathcal{H}$, that is $\|\ \|$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon\colon \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has $\nu(\{x \in \mathcal{H} \mid \|P(x)\| > \epsilon\}) < \epsilon$, where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $\|\ \|$ is weaker than $|\ |$. Denote by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $\|\ \|$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space.*[48, 62] Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$ and each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x\rangle\, e_i$$

for some orthonormal $e_1,\ldots, e_n \in \mathscr{H}$, the stochastic extension $\tilde{P}$ of P on $\mathscr{B}$ is well defined by

$$\tilde{P}(\cdot)=\sum_{i=1}^{n} n(e_i)(\cdot)\, e_i.$$

Given a function $f\colon \mathscr{H} \to \mathscr{B}_1$, where $(\mathscr{B}_1, \|\ \|_{\mathscr{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathscr{B}$ exists if the functions $f\circ \tilde{P}\colon \mathscr{B}\to\mathscr{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathscr{H}$. Let $A\colon \mathscr{H}\times\mathscr{H}\times\mathscr{H}\times\mathscr{H}\to\mathbb{R}$ be a completely symmetric positive covariant tensor operator on $\mathscr{H}$ such that the map $V\colon \mathscr{H}\to\mathbb{R}^+$, $x\mapsto V(x)\equiv A(x,x,x,x)$ is continuous in the $\|\ \|$ norm. As a consequence V is continuous in the $|\ |$-norm, moreover it can be extended by continuity to a random variable $\bar{V}$ on $\mathscr{B}$, with $\bar{V}|_{\mathscr{H}}=V$. Moreover given a self-adjoint trace class operator $B\colon \mathscr{H}\to\mathscr{H}$, the quadratic form on $\mathscr{H}\times\mathscr{H}$:

$$x\in\mathscr{H}\mapsto\langle x, Bx\rangle$$

can be extended to a random variable on $\mathscr{B}$, denoted again by $\langle\,\cdot\,, B\,\cdot\,\rangle$. In this setting one can prove the following generalization of Theorem 6.[21]

Theorem 7. Let B be self-adjoint trace class, $(I-B)$ strictly positive, $\lambda\leqslant 0$ and $f\in\mathscr{F}(\mathscr{H})$, $f\equiv\hat{\mu}_f$, and let us suppose that the bounded variation measure μ_f satisfies the following assumption

$$\int_{\mathscr{H}} e^{\frac{\hbar}{4}\langle k, (I-B)^{-1}k\rangle}\, |\mu_f|\,(dk)<+\infty. \tag{31}$$

Then the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathscr{H}}} e^{\frac{i}{2\hbar}\langle x, (I-B)\,x\rangle}\, e^{-i\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)\, dx \tag{32}$$

exists and is given by:

$$\int_{\mathscr{H}} \mathbb{E}[e^{in(k)(\omega)}\, e^{i\pi/4}\, e^{\frac{1}{2\hbar}\langle\omega, B\omega\rangle} e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}]\ \mu_f(dk).$$

It is also equal to:

$$\mathbb{E}[e^{i\frac{\lambda}{\hbar}\bar{V}(\omega)}e^{\frac{1}{2\hbar}\langle\omega, B\omega\rangle}f(e^{i\pi/4}\omega)] \tag{33}$$

$\mathbb{E}$ denotes the expectation value with respect to the Gaussian measure μ on $\mathscr{B}$.

Such a theory allows an extension of the class of potentials for which an infinite dimensional oscillatory integral representation of the solution of the corresponding Schrödinger equation can be defined. Let us consider the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi = H\psi \tag{34}$$

on $L^2(\mathbb{R}^d)$ for an anharmonic oscillator Hamiltonian H of the following form:

$$H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}x\Omega^2 x + \lambda C(x, x, x, x), \tag{35}$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$, Ω is a positive symmetric $d\times d$ matrix, $\lambda \geqslant 0$ a positive constant. It is well known, see ref. 69, that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. We are going to show the way to give mathematical meaning to the "Feynman path integral" representation of the solution of Eq. (34):

$$\psi(t, x) = \text{``}\int_{\gamma(0)=x} e^{\frac{i}{\hbar}\int_0^t \frac{\dot{\gamma}(s)^2}{2}ds - \frac{i}{\hbar}\int_0^t [\frac{1}{2}\gamma(s)\,\Omega^2\gamma(s) + \lambda C(\gamma(s), \gamma(s), \gamma(s), \gamma(s))]\,ds}\,\psi_0(\gamma(t))\,D\gamma\text{''},$$

as the analytic continuation (in the parameter λ) of an infinite dimensional generalized oscillatory integral on a suitable Hilbert space.

Let us consider the Cameron-Martin space H_t, that is the Hilbert space of absolutely continuous paths γ: $[0, t] \to \mathbb{R}^d$, with $\gamma(0) = 0$ and inner product $\langle\gamma_1, \gamma_2\rangle = \int_0^t \dot{\gamma}_1(s)\,\dot{\gamma}_2(s)\,ds$. The cylindrical Gaussian measure on H_t with covariance operator the identity extends to a σ-additive measure on the Wiener space $C_t = \{\omega \in C([0, t]; \mathbb{R}^d) \mid \gamma(0) = 0\}$: the Wiener measure W. (i, H_t, C_t) is an abstract Wiener space.

Let us consider moreover the Hilbert space $\mathscr{H} = \mathbb{R}^d \times H_t$, and the Banach space $\mathscr{B} = \mathbb{R}^d \times C_t$ endowed with the product measure $N(dx) \times W(d\omega)$, N being the Gaussian measure on $\mathbb{R}^d$ with covariance equal to the $d \times d$ identity matrix. $(i, \mathscr{H}, \mathscr{B})$ is an abstract Wiener space.

Let us consider two vectors $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$.We are going to define the following infinite dimensional oscillatory integral on $\mathcal{H}$:

$$\text{“}\int_{\mathbb{R}^d \times H_t} \bar{\phi}(x)\, e^{\frac{i}{2\hbar}\int_0^t \dot{\gamma}(s)^2\, ds} e^{-\frac{i}{2\hbar}\int_0^t [(\gamma(s)+x)\,\Omega^2(\gamma(s)+x)\, ds} \cdot e^{\frac{i\lambda}{\hbar} C(\gamma(s)+x,\, \gamma(s)+x,\, \gamma(s)+x,\, \gamma(s)+x)\, ds} \psi_0(\gamma(t)+x)\, dx\, D\gamma\text{”}. \tag{36}$$

Let us consider the operator B: $\mathcal{H} \to \mathcal{H}$ given by:

$$(x, \gamma) \longrightarrow (y, \eta) = B(x, \gamma),$$

$$y = t\Omega^2 x + \Omega^2 \int_0^t \gamma(s)\, ds, \qquad \eta(s) = \Omega^2 x\left(ts - \frac{s^2}{2}\right) - \int_0^s \int_t^u \Omega^2 \gamma(r)\, dr\, du \tag{37}$$

and the fourth order tensor operator A given by:

$$A((x_1, \gamma_1), (x_2, \gamma_2), (x_3, \gamma_3), (x_4, \gamma_4)) = \int_0^t C(\gamma_1(s)+x_1, \gamma_2(s)+x_2, \gamma_3(s)+x_3, \gamma_4(s)+x_4)\, ds. \tag{38}$$

Let us consider moreover the function f: $\mathcal{H} \to \mathbb{C}$ given by

$$f(x, \gamma) = (2\pi i\hbar)^{d/2}\, e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x)\, \psi_0(\gamma(t)+x) \tag{39}$$

with this notation expression (36) can be written in the following form:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+|\gamma|^2)} e^{-\frac{i}{2\hbar}\langle (x, \gamma), B(x, \gamma)\rangle} e^{-\frac{i\lambda}{\hbar} A((x, \gamma), (x, \gamma), (x, \gamma), (x, \gamma))} f(x, \gamma)\, dx\, d\gamma. \tag{40}$$

In the following we will denote by Ω_i, $i = 1,\ldots, d$, the eigenvalues of the matrix Ω.

Theorem 8. Let us assume that $\lambda \leqslant 0$, and that for each $i = 1,\ldots, d$ the following inequalities are satisfied

$$\Omega_i t < \frac{\pi}{2}, \qquad 1 - \Omega_i \tan(\Omega_i t) > 0. \tag{41}$$

Let $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. Let μ_0 be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_0 = \psi_0$. Let μ_ϕ be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$. Assume in addition that the measures μ_0, μ_ϕ satisfy the following assumption:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)\,x} e^{(y+\cos(\Omega t)^{-1}x)(1-\Omega\tan(\Omega t))^{-1}(y+\cos(\Omega t)^{-1}x)}\,|\mu_0|\,(dx)\,|\mu_\phi|\,(dy) < \infty. \tag{42}$$

Then the function $f\colon \mathcal{H} \to \mathbb{C}$, given by (39) is the Fourier transform of a bounded variation measure μ_f on $\mathcal{H}$ satisfying

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),\,(I-B)^{-1}(y,\eta)\rangle}\,|\mu_f|\,(dy\,d\eta) < \infty \tag{43}$$

(B being given by (37)) and the infinite dimensional oscillatory integral (40) is well defined and is given by:

$$\int_{\mathbb{R}^d\times H_t}\left(\int_{\mathbb{R}^d\times C_t} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}\,n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\,\omega(s)+x)\,\Omega^2(\sqrt{\hbar}\,\omega(s)+x)\,ds}\right.$$
$$\left.\cdot\, e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x)\,ds}\,W(d\omega)\,\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}\,dx\right)\mu_f(dy\,d\gamma). \tag{44}$$

This is also equal to

$$(i)^{d/2}\int_{\mathbb{R}^d\times C_t} e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x,\sqrt{\hbar}\,\omega(s)+x)\,ds}$$
$$\cdot\, e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\,\omega(s)+x)\,\Omega^2(\sqrt{\hbar}\,\omega(s)+x)\,ds}\,\bar{\phi}(e^{i\pi/4}x)\,\psi_0(e^{i\pi/4}\sqrt{\hbar}\,\omega(t)+e^{i\pi/4}x)\,W(d\omega)\,dx. \tag{45}$$

The oscillatory integral (40) can heuristically be written in the following form:

$$(\phi,\psi(t)) = \text{“}\int_{\mathbb{R}^d}\bar{\phi}(x)\int_{\{\gamma\,|\,\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))\,D\gamma\,dx\text{”}$$

and interpreted as a rigorous realization of the Feynman path integral representing the inner product between the vector $\phi \in L^2(\mathbb{R}^d)$ and the solution of the Schrödinger equation (34) with initial datum ψ_0. We remark that the infinite dimensional oscillatory integral (40) is well defined only if $\lambda \leqslant 0$, but its expressions in terms of the absolutely convergent integrals (44) and (45) are well defined also for each $\lambda \in \mathbb{R}$. In fact one can verify directly that the absolutely convergent integrals (44) and (45) are analytic functions of the complex variable λ if $\mathrm{Im}(\lambda) > 0$, continuous in $\mathrm{Im}(\lambda) = 0$ and coinciding with (40) if $\lambda \leqslant 0$. Moreover one can prove that when $\lambda \geqslant 0$ the Gaussian integrals (44) and (45) represent the inner product $\langle \phi, \psi(t) \rangle$, where $\psi(t)$ is the solution of the Schrödinger equation.

Theorem 9. Let $\phi, \psi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfy assumption (42). Then (44) and (45) represent, for $\lambda \geqslant 0$, the scalar product between ϕ and the solution of the Schrödinger equation (34).

3. A NOTE ON PROBABILISTIC METHODS IN QUANTUM FIELD THEORY

Extensions of the methods discussed in Sections 1 and 2 to infinite dimensional state spaces have also been studied, particularly in relations to applications to quantum field theory.

As for the Feynman path integrals we simply mention the rigorous construction of the Chern–Simons model of gauge fields in 3 space-time dimensions and the corresponding rigorous computation of topological invariants using the theory of infinite dimensional oscillatory integrals (Fresnel integrals), see refs. 15, 27, 28, 63, and references therein.

As for the heat semigroup evolutions, associated infinite dimensional processes and their connections to Euclidean quantum fields let us just mention that the measure which plays a corresponding role to the one mentioned in Section 1, $e^{-\int_0^\tau V(w(s))\,ds} P^x(dw)$, is (for Euclidean scalar quantum fields over the Euclidean space-time $\mathbb{R}^d$)

$$\mu^v(dw) = \text{``}Z^{-1} e^{-\lambda \int_{\mathbb{R}^d} v(w(y))\,dy} \mu(dw)\text{''} \tag{46}$$

where $Z \equiv$ "$\int e^{-\lambda \int_{\mathbb{R}^d} v(w(y))\,dy} \mu(dw)$", $\lambda > 0$, $v\colon \mathbb{R} \to \mathbb{R}$ being an "interaction density," μ being Nelson's free field measure (describing free Euclidean quantum fields over $\mathbb{R}^d$ of mass $m > 0$). w is the Euclidean coordinate process, representing the Euclidean quantum field (over $\mathbb{R}^d$).

For $d = 2$, μ^v can be given a rigorous meaning (for suitable v), see, e.g., refs. 19, 45, 49, and 71.

μ^v was given a further meaning as "equilibrium measure" for a stochastic evolution equation, the "stochastic quantization equation," in work by Jona-Lasinio and coworkers[(56)] (see also refs. 22–25). This has given also a considerable momentum to the theory of stochastic partial differential equations, see also, for instance, refs. 58 and 59. μ itself can be looked upon as "convolution transform" of the Gaussian white noise measure μ_{GWN},[49] i.e.,

$$\mu = \mu_{\text{GWN}}((\sqrt{-\Delta + m^2})^{-1}.), \tag{47}$$

Δ being the Laplacian in $L^2(\mathbb{R}^d)$ (see refs. 10 and 49).

Models constructed by replacing in (46), (47) μ_{GWN} by μ_{PWN}, with μ_{PWN} a Poisson-type white noise measure have been constructed in recent years and shown to satisfy, together with their vector-valued analogues, all Morchio-Strocchi axioms for indefinite metric quantum fields, see refs. 9 and 11, and references therein.

Relativistic models with local interaction and non-trivial scattering (even in 4-space-time dimensions!) have been constructed.[8] For recent work on extending these models to curved space-times see ref. 47.

New connections with models of (classical) statistical mechanics (of particles) have emerged from this work, see refs. 9 and 46.

Previous work on the use of Poisson measures in the study of quantum models is in ref. 39.

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